

## Decision Making and the Value of Forecasts in a Generalized Model of the Cost-Loss Ratio Situation

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### ABSTRACT

Meteorologists have devoted considerable attention to studies of the use and value of forecasts in a simple two-action, two-event decision-making problem generally referred to as the cost-loss ratio situation. An  $N$ -action,  $N$ -event generalization of the standard cost-loss ratio situation is described here, and the expected value of different types of forecasts in this situation is investigated. Specifically, expressions are developed for the expected expenses associated with the use of climatological, imperfect, and perfect information, and these expressions are employed to derive formulas for the expected value of imperfect and perfect forecasts. The three-action, three-event situation is used to illustrate the generalized model and the value-of-information results, by considering examples based on specific numerical values of the relevant parameters. Some possible extensions of this model are briefly discussed.

### 1. Introduction

Meteorologists have devoted considerable attention to studies of the use and value of weather information in the context of a simple, static decision-making problem commonly referred to as the "cost-loss ratio situation" (e.g., Thompson, 1952, 1962; Thompson and Brier, 1955; Murphy, 1977). In essence, this problem involves a decision maker who must decide whether or not to protect an activity or operation in the face of uncertainty as to whether or not adverse weather will occur. The decision maker is assumed to use weather information to guide the choice of an optimal course of action. Despite its obvious simplicity, this "standard" cost-loss ratio situation represents a useful prototype of many weather-information-sensitive, decision-making problems.

Notwithstanding the simple generality and usefulness of the standard cost-loss ratio situation, it suffers from at least two important shortcomings as a "model" of many decision-making problems. First, such problems frequently are repetitive in nature and decisions made in these situations generally are related over time. Repeated application of the static cost-loss ratio model obviously would be inappropriate in such situations. A forthcoming paper (Murphy *et al.*, 1985) describes a dynamic model of the standard two-action, two-event cost-loss ratio situation and examines the value of different types of information in this situation. Second, many weather-information-sensitive problems involve situations in which the decision maker is concerned with more than two actions and more than two events. The purposes of

this paper are to describe a generalized model of the standard situation involving  $N$  actions and  $N$  events and to investigate the value of information in the context of a static application of this model.

The generalized model, which involves  $N$  levels of protection and  $N$  degrees of adverse weather, is described in Section 2. This model was first formulated by Epstein (1969), who used it solely as a framework for the development of the ranked probability score. Section 2 also contains a description of the procedure for choosing the optimal action in this situation. Expressions for the expected expense associated with the use of climatological, imperfect, and perfect information in the context of this model are formulated in Section 3, and these expressions are employed to derive formulas for determining the expected value of imperfect and perfect forecasts. Section 4 contains an illustration of the generalized model and the value-of-information results in the three-action, three-event situation. Finally, Section 5 consists of a brief discussion and conclusion.

### 2. The $N$ -action, $N$ -event cost-loss ratio situation

#### a. Basic decision-making problem

In this generalization of the standard two-action, two-event cost-loss ratio situation, it is assumed that the decision maker must select one of  $N$  admissible actions  $P_i$  ( $i = 1, \dots, N$ ). These actions represent  $N$  levels of protection, ranging from full protection ( $P_1$ ) to no protection ( $P_N$ ). It is also assumed that the relevant weather conditions are described in terms of  $N$  mutually exclusive and collectively exhaustive

events  $W_j$  ( $j = 1, \dots, N$ ). These events represent  $N$  degrees of adverse weather, ranging from completely adverse weather ( $W_1$ ) to no adverse weather ( $W_N$ ).

This  $N$ -action,  $N$ -event situation involves  $N^2$  consequences, and the "impacts" of these consequences on the decision maker are measured in terms of costs of protection and/or losses which may be incurred if protection is inadequate. The cost of protection is assumed to decrease linearly from  $C$ , the cost of full protection ( $P_1$ ), to zero, the cost of no protection ( $P_N$ ). Thus, the cost of protection associated with action  $P_i$  is  $[(N-i)/(N-1)]C$ . Moreover, when action  $P_i$  is taken, the activity is assumed to be completely protected against adverse weather up to and including event  $W_i$ . For combinations of actions  $P_i$  and events  $W_j$  for which  $i > j$ , a loss is incurred and this loss is assumed to increase linearly from  $L/(N-1)$ , for  $P_i$  and  $W_j$  with  $i = j+1$ , to  $L$ , for  $P_N$  and  $W_1$ . In summary, if  $E_{ij}$  denotes the expense associated with action  $P_i$  and event  $W_j$ , then

$$E_{ij} = \begin{cases} [(N-i)/(N-1)]C & \text{if } i \leq j \\ [(N-i)C + (i-j)L]/(N-1) & \text{if } i > j \end{cases} \quad (1)$$

( $i, j = 1, \dots, N$ ). Examination of  $E_{ij}$  in (1) reveals that the standard two-action, two-event cost-loss ratio situation is a special case of this more general problem when  $N = 2$ . It should also be noted that the assumption of admissibility of the  $N$  actions implies that  $0 < C < L < \infty$ .

It is convenient to redefine the  $E_{ij}$  in (1) as expenses *per unit loss*. If these "standardized" expenses are denoted by  $E'_{ij}$ , then

$$E'_{ij} = \begin{cases} [(N-i)/(N-1)](C/L) & \text{if } i \leq j \\ [(N-i)(C/L) + (i-j)]/(N-1) & \text{if } i > j \end{cases} \quad (2)$$

( $i, j = 1, \dots, N$ ). We use the standardized expenses  $E'_{ij}$  in (2) throughout the remainder of this paper.

#### b. Optimal actions

As in the case of the standard cost-loss ratio situation, the decision maker is assumed to choose the action that minimizes his/her expected expense. The expected expense of an action is the probability-weighted average of the expenses (i.e., costs and/or losses) associated with that action. This criterion is equivalent to maximizing expected utility under the assumption that the decision maker's utility function is linear in monetary expense.

Let  $r_j$  denote the decision maker's probability of event  $W_j$ , where  $r_j \geq 0$  and

$$\sum_{j=1}^N r_j = 1.$$

Thus, if  $EE(P_i)$  denotes the expected expense associated with action  $P_i$ , then

$$EE(P_i) = \sum_{j=1}^N r_j E'_{ij}, \quad (3)$$

or, from (2),

$$EE(P_i) = [(N-i)/(N-1)](C/L) + [1/(N-1)] \sum_{j=1}^i (i-j)r_j. \quad (4)$$

As noted above, the decision maker prefers action  $P_i$  to action  $P_{i+1}$  if  $EE(P_i) < EE(P_{i+1})$ . Using (4), it is relatively easy to show that this condition holds when

$$\sum_{j=1}^i r_j > C/L.$$

Thus, the decision maker prefers action  $P_i$  to the other  $N-1$  actions when

$$\sum_{j=1}^{i-1} r_j < C/L < \sum_{j=1}^i r_j$$

(see Epstein, 1969).

### 3. Forecasts: Expense and value expressions

#### a. Types of forecasts

Three types of information or forecasts are considered in this paper: 1) imperfect forecasts; 2) climatological information; and 3) perfect information. In order to describe these types of information, it is necessary to introduce the following notation:

1) Events and climatological probabilities:

$W_j$  ( $j = 1, \dots, N$ ) denotes the  $N$  degrees of adverse weather (or events), with  $W_j = 1$  if the  $j$ th event occurs and  $W_j = 0$  otherwise;

$p_j = \Pr(W_j = 1)$  is the climatological (or prior) probability of the  $j$ th event

$$(p_j \geq 0, \sum_j p_j = 1; j = 1, \dots, N).$$

2) Forecasts:

$F_l$  ( $l = 1, \dots, N$ ) denotes the  $N$  possible (categorical) forecasts of the degrees of adverse weather, with  $F_l = 1$  if the  $l$ th event is forecast and  $F_l = 0$  otherwise;  $p_{jl} = \Pr(W_j = 1 | F_l = 1)$  is the conditional (or posterior) probability of the  $j$ th event given a forecast of the  $l$ th event

$$(p_{jl} \geq 0, \sum_j p_{jl} = 1; j, l = 1, \dots, N);$$

$\pi_l = \Pr(F_l = 1)$  is the (predictive) probability of a forecast of the  $l$ th event

$$(\pi_l \geq 0, \sum_l \pi_l = 1; l = 1, \dots, N).$$

It is important to recognize that certain relationships exist among the three sets of probabilities defined above. Specifically, according to the definition of conditional probability,

$$p_j = \sum_{l=1}^N \pi_l p_{jl}, \quad j = 1, \dots, N. \quad (5)$$

The quality of the imperfect forecasts is described completely by the conditional probabilities  $p_{jl}$  ( $j, l = 1, \dots, N$ ). These forecasts can be considered to be categorical (i.e., nonprobabilistic) forecasts of the degrees of adverse weather properly "calibrated" according to past performance. This formulation of imperfect forecasts also can be viewed as a special case of probability forecasts in which only  $N$  sets of probabilities are used.

Climatological information corresponds to the limiting case of imperfect forecasts for which  $p_{jl} = p_j$  for all  $l$  ( $l = 1, \dots, N$ ). Perfect information, on the other hand, corresponds to the limiting case of imperfect forecasts for which  $p_{jl} = 1(0)$  if  $j = (\neq) l$  and necessarily  $p_l = \pi_l$  ( $j, l = 1, \dots, N$ ). Climatological and perfect information are of particular interest because they can be considered to represent lower and upper bounds, respectively, on the quality of imperfect forecasts.

#### b. Expense

The expense and value expressions derived in this paper are based on a decision-analytic approach to value-of-information assessment (e.g., Winkler *et al.*, 1983). Moreover, it is assumed here that the decision maker adopts the information or forecasts (whether climatological, imperfect, or perfect) as the sole basis for choosing the optimal action. In this context, the expected expense associated with the use of imperfect forecasts can be determined in the following manner. For a particular forecast  $F_l$  and a particular level of protection  $P_k$ , the expected expense associated with the probabilities  $p_{jl}$  is  $E_k(l)$ , where

$$E_k(l) = \sum_{j=1}^N p_{jl} E'_{kj}. \quad (6)$$

Since the decision maker is concerned with minimizing expected expense, he/she will choose the level of protection,  $P_{k^*}$  say, that minimizes  $E_k(l)$  in (6) (the optimal level of protection, in general, will depend upon the forecast  $F_l$ ). Thus,

$$E_{k^*}(l) = \min_k \left( \sum_{j=1}^N p_{jl} E'_{kj} \right). \quad (7)$$

Finally, since the expected expense  $E_{k^*}(l)$  in (7) is a function of the forecast  $F_l$ , it is necessary to "average out"  $E_{k^*}(l)$  using the predictive probabilities  $\pi_l$

( $l = 1, \dots, N$ ). If this average expected expense is denoted by EF, then

$$EF = \sum_{l=1}^N \pi_l E_{k^*}(l), \quad (8)$$

or, from (7),

$$EF = \sum_{l=1}^N \pi_l \min_k \left( \sum_{j=1}^N p_{jl} E'_{kj} \right). \quad (9)$$

It is also possible to rewrite (9) in terms of the basic costs and losses by employing the results related to optimal actions presented in Section 2b. Thus, from (4),

$$EF = \sum_{l=1}^N \pi_l \{ [(N - k_l^*) / (N - 1)] (C/L) + [1 / (N - 1)] \sum_{j=1}^{k_l^*} (k_l^* - j) p_{jl} \}, \quad (10)$$

in which the value of the index  $k_l^*$  is determined by the inequality

$$\sum_{j=1}^{k_l^*-1} p_{jl} < C/L < \sum_{j=1}^{k_l^*} p_{jl} \quad (11)$$

( $l = 1, \dots, N$ ).

Let EC denote the expected expense associated with the use of climatological information. Since climatological information represents the limiting case of imperfect forecasts in which  $p_{jl} = p_j$  for all  $l$  ( $l = 1, \dots, N$ ), EC can be expressed as follows [from (9)]:

$$EC = \min_i \sum_{j=1}^N p_j E'_{ij}, \quad (12)$$

or, from (2),

$$EC = [(N - i) / (N - 1)] (C/L) + [1 / (N - 1)] \sum_{j=1}^i (i - j) p_j, \quad (13)$$

in which the index  $i$  is defined by the inequality

$$\sum_{j=1}^{i-1} p_j < C/L < \sum_{j=1}^i p_j. \quad (14)$$

Let EP denote the expected expense associated with the use of perfect information. Since perfect information represents the limiting case of imperfect forecasts in which  $p_{jl} = 1(0)$  if  $j = (\neq) l$  and  $\pi_l = p_l$ , EP can be expressed as follows [from (9) and (2)]:

$$EP = [1 / (N - 1)] (C/L) \sum_{j=1}^N (N - j) p_j. \quad (15)$$

A natural ordering exists among the expected ex-

penses associated with these three types of information. Specifically, it can be shown that

$$EP \leq EF \leq EC. \quad (16)$$

That is, the expected expense associated with perfect information is less than or equal to the expected expense associated with imperfect forecasts, and the latter is in turn less than or equal to the expected expense associated with climatological information. This result is a special case of a fundamental theorem of Blackwell (1953) concerning the comparative value of experiments.

### c. Value

Since the value of information is a comparative concept, it is natural to define the value of the forecasts of interest here relative to climatological information. That is, EC serves as a convenient "zero point" (or origin) of the value scale [see (16)]. Thus, if VP and VF denote the expected value of perfect and imperfect forecasts, respectively, then

$$VP = EC - EP, \quad (17)$$

$$VF = EC - EF. \quad (18)$$

Moreover, from (16),

$$0 \leq VF \leq VP. \quad (19)$$

That is, the expected value of imperfect forecasts is always nonnegative, but it cannot exceed the expected value of perfect information.

It is sometimes useful to compare the magnitudes of VF and VP to determine the extent to which the expected value of the forecasts of interest "approaches" the expected value of perfect information. For this purpose, it may be desirable to compute the efficiency (or effectiveness)  $\epsilon$  of the forecasts, where

$$\epsilon = VF/VP. \quad (20)$$

Note that, from (19),  $0 \leq \epsilon \leq 1$ .

## 4. Illustration: The three-action, three-state situation

### a. Decision-making problem

The basic tableau for the  $N$ -action,  $N$ -event decision-making problem is depicted in Table 1 for the case  $N = 3$ . Standardized expenses  $E'_{ij}$  ( $i, j = 1, 2, 3$ ) corresponding to the various combinations of actions  $P_i$  and events  $W_j$  were obtained from (2) in Section 2a. The probabilities  $p_j$  ( $j = 1, 2, 3$ ) can be considered to represent the climatological (or prior) probabilities of the respective events.

To determine the optimal action for various values of the probabilities  $p_j$  ( $j = 1, 2, 3$ ) (or any other probabilities, for that matter), it is necessary to compare the expected expenses associated with the actions. As noted in Section 2, these expected expenses are

TABLE 1. The basic tableau for the  $N$ -action,  $N$ -event cost-loss ratio situation when  $N = 3$ , including standardized expenses  $E'_{ij}$  and climatological probabilities  $p_j$  ( $i, j = 1, 2, 3$ ).

	Events		
	$W_1$	$W_2$	$W_3$
Actions			
$P_1$	$C/L$	$C/L$	$C/L$
$P_2$	$(C/2L) + (1/2)$	$C/2L$	$C/2L$
$P_3$	1	$1/2$	0
Probabilities	$p_1$	$p_2$	$p_3$

the probability-weighted averages of the expenses (costs and/or losses) associated with the respective actions. Thus, if we denote these expected expenses by  $EE(P_i)$  ( $i = 1, 2, 3$ ), then

$$EE(P_1) = C/L, \quad (21)$$

$$EE(P_2) = \frac{1}{2} (C/L) + \frac{1}{2} p_1, \quad (22)$$

$$EE(P_3) = p_1 + \frac{1}{2} p_2 \quad (23)$$

[see (4)]. Pairwise comparison of  $EE(P_1)$ ,  $EE(P_2)$  and  $EE(P_3)$  reveals that, as expected (see Section 2b),  $P_1$  is the optimal action when  $p_1 > C/L$ ,  $P_2$  is the optimal action when  $p_1 < C/L < p_1 + p_2$ , and  $P_3$  is the optimal action when  $p_1 + p_2 < C/L$ .

Given the value of the cost-loss ratio  $C/L$ , the inequalities in the previous paragraph specify sets of values of the probabilities  $p_j$  ( $j = 1, 2, 3$ ) for which each action is optimal. It is possible to depict these sets within a geometrical framework by recognizing that the set of all possible probabilities constitutes an equilateral triangle in the three-event situation. Such a triangle is presented in Fig. 1, in which a barycentric coordinate system is used to describe the correspondence between a set of probability values ( $p_1, p_2, p_3$ ) and a point in (or on) the triangle. In this coordinate system, the triangle is assumed to have unit altitude, and distances along the axes are measured from the sides of the triangle to the opposite vertices. Thus, the vertices represent the three possible categorical forecasts (1, 0, 0), (0, 1, 0), and (0, 0, 1). The sets of points, or regions, of the triangle for which  $P_i$  ( $i = 1, 2, 3$ ) is the optimal action are indicated in Fig. 1, under the assumption that  $C/L = 0.3$ . As expected,  $P_1$  is optimal if  $p_1 > 0.3$ ,  $P_2$  is optimal if  $p_1 < 0.3 < p_1 + p_2$ , and  $P_3$  is optimal if  $p_1 + p_2 < 0.3$ . The boundaries between these regions are lines of equal expected expense for the corresponding pair of actions (or lines of indifference for the decision maker), determined by equating the respective expected expenses.

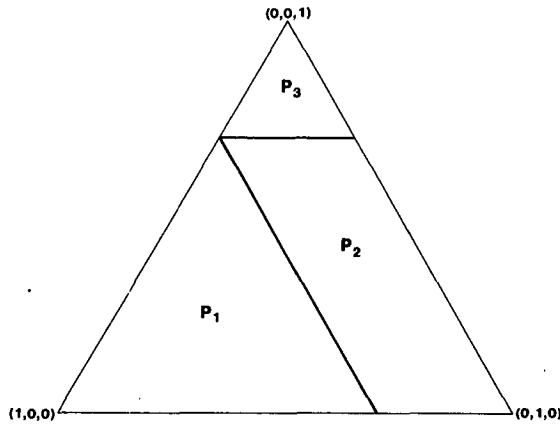


FIG. 1. The sets of climatological probabilities ( $p_1, p_2, p_3$ ) for which action  $P_i$  ( $i = 1, 2, 3$ ) is preferred in a three-action, three-event situation in which  $C/L = 0.3$ . Each point in (or on) the equilateral triangle corresponds to a set of possible climatological probabilities (see text for additional details).

### b. Value of forecasts

In this section we examine the expected value of perfect and imperfect forecasts in the three-action, three-event situation and present some numerical results for specific values of the relevant parameters (namely, the three types of probabilities and the cost-loss ratio). The expression for the expected value of perfect information VP in this situation can be written as

$$VP = [(3-i)/2](C/L) + \frac{1}{2} \sum_{j=1}^i (i-j)p_j - \frac{1}{2} (C/L)(2p_1 + p_2), \quad (24)$$

in which the numerical value of the index  $i$  is determined by the inequality (14) [see (13), (15) and (17)]. Thus, three cases can be identified:

Case 1:  $0 < C/L < p_1$ ,  $P_1$  is the optimal action, and

$$VP = (C/L) \left[ 1 - \frac{1}{2} (2p_1 + p_2) \right]. \quad (25)$$

Case 2:  $p_1 < C/L < p_1 + p_2$ ,  $P_2$  is the optimal action, and

$$VP = \frac{1}{2} [(1 - 2p_1 - p_2)(C/L) + p_1]. \quad (26)$$

Case 3:  $p_1 + p_2 < C/L < 1$ ,  $P_3$  is the optimal action, and

$$VP = \frac{1}{2} (2p_1 + p_2) [1 - (C/L)]. \quad (27)$$

The values of VP for these cases are depicted within the geometrical framework of the equilateral triangle (i.e., as a function of the climatological probabilities  $p_1, p_2$  and  $p_3$ ) in Fig. 2, under the assumption that  $C/L = 0.3$ . Broken lines in this figure connect points

with equal values of VP. Small values of VP occur near the three vertices. Obviously, if prior (i.e., climatological) information "places" the decision maker at a point near a vertex, then perfect information can be of relatively little additional value. Large values (local maxima) of VP occur near (at) the boundaries between the regions in which the respective actions are optimal. This result too is intuitively reasonable, since the decision maker's choice of an optimal action is most "sensitive" in these situations (and it is changes in optimal actions that determine the value of forecasts, whether they are perfect or imperfect). In this regard, it is of interest to note that the global maximum of VP (0.21) occurs at the intersection of the two boundaries [ $(p_1, p_2, p_3) = (0.30, 0.00, 0.70)$ ]. These results are consistent with the analogous results in the standard two-action, two-event cost-loss ratio situation, in which VP = 0 when  $p_1 = 0$  or 1, and VP attains its maximum value when  $p_1 = C/L$  (see Winkler and Murphy, 1985). In a specific numerical example to be presented below, we will assume that the climatological probabilities are (0.1, 0.3, 0.6). This point is denoted by  $p^*$  in Fig. 2, and VP = 0.125 for these probabilities [see (24)].

The expression for the expected value of imperfect forecasts VF in the three-action, three-event situation can be written as

$$VF = [(3-i)/2](C/L) + \frac{1}{2} \sum_{j=1}^i (i-j)p_j - \sum_{l=1}^3 \pi_l \{ [(3-k_l^*)/2](C/L) + \frac{1}{2} \sum_{j=1}^{k_l^*} (k_l^* - j)p_{jl} \} \quad (28)$$

in which the values of the indices  $i$  and  $k_l^*$  ( $l = 1, 2, 3$ ) are determined by the inequalities (14) and (11), respectively [see (10), (13) and (18)]. Since each of these indices can take on three values, 81 ( $=3^4$ ) different cases may exist! In this regard, however, it should be recalled that the three types of probabilities are related as follows:

$$p_j = \sum_{i=1}^3 \pi_i p_{ji} = \pi_1 p_{j1} + \pi_2 p_{j2} + \pi_3 p_{j3} \quad (29)$$

( $j = 1, 2, 3$ ) [see (5)].

Instead of considering all or some of these cases, we will examine the expected value of two hypothetical forecasting systems, A and B say, whose quality is characterized by the conditional probabilities  $p_{jl}^A$  and  $p_{jl}^B$  presented in Tables 2a and 2b, respectively ( $j, l = 1, 2, 3$ ). In "formulating" these two sets of probabilities, it has been assumed that  $\pi_1 = p_1 = 0.1$ ,  $\pi_2 = p_2 = 0.3$ , and  $\pi_3 = p_3 = 0.6$ . The assumption that  $\pi_j = p_j$  ( $j = 1, 2, 3$ ) is equivalent to assuming that the forecasts are unbiased in-the-large; that is, that the forecast frequencies of the events are equal to the observed frequencies of the events. Consistent with

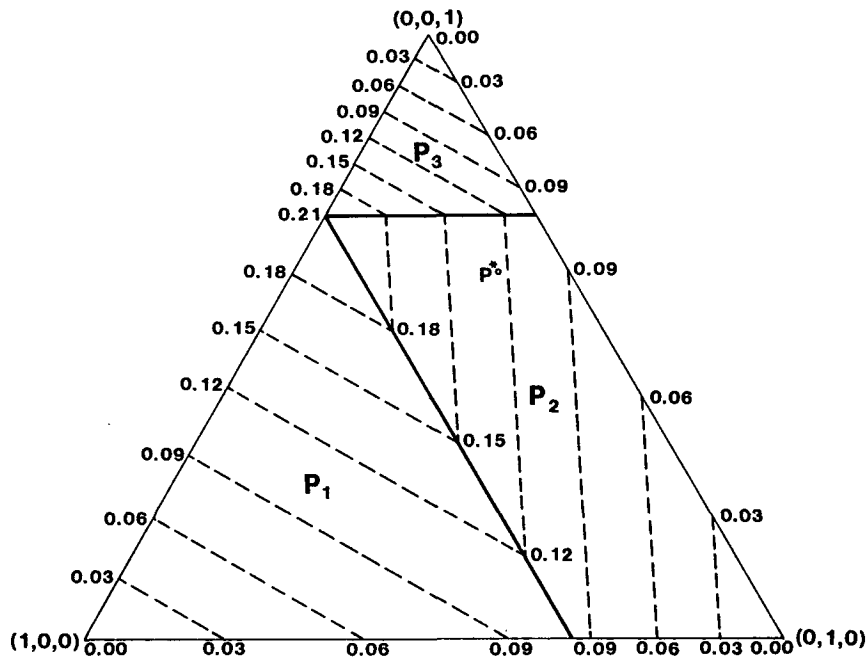


FIG. 2. The value of perfect information VP in a three-action, three-event situation in which  $C/L = 0.3$ . The point  $p^* = (0.1, 0.3, 0.6)$  represents a particular set of climatological probabilities (see text for additional details).

previous numerical examples, we also assume that  $C/L = 0.3$ .

In evaluating VF for Forecasting Systems A and B using (28), we must determine the numerical values of the indices  $i$  and  $k_l^*$  ( $l = 1, 2, 3$ ). For both systems,  $i = 2$ ,  $k_1^* = 1$ ,  $k_2^* = 2$ , and  $k_3^* = 3$  [see Table 2 and (11)]. Then, from (28), we find that  $VF^A = 0.053$  and  $VF^B = 0.065$ . Since  $VP = 0.125$ , it follows from (20) that  $\epsilon^A = 0.424$  and  $\epsilon^B = 0.520$ . Thus, the efficiency of Forecasting System B is almost 10% greater than that of Forecasting System A.

It also may be of interest to compare this difference in efficiency with the difference in quality (e.g., ac-

curacy) of the two forecasting systems. The ranked probability score (RPS) (Epstein, 1969) is obviously a reasonable measure of accuracy in this situation, where

$$RPS = \sum_{l=1}^N \sum_{j=1}^N \pi_l p_{jl} \sum_{k=1}^N \left( \sum_{h=1}^k p_{hl} - \delta_k \right)^2, \quad (30)$$

in which  $\delta_k = 1$  if  $k \geq j$  and  $\delta_k = 0$  otherwise. Using (30), we find that  $RPS^A = 0.220$  and  $RPS^B = 0.192$ . Thus, a 12–13% increase in accuracy is reflected in a 9–10% increase in efficiency for the forecasting systems and decision-making situation considered here.

## 5. Discussion and conclusion

An  $N$ -action,  $N$ -event decision-making problem (or “model”) that is a generalization of the familiar two-action, two-event cost–loss ratio situation has been described. Obviously, many such generalizations are possible, but this particular model appears to represent a simple yet useful prototype of these generalizations. In this regard, the expenses (i.e., costs and/or losses) in this problem are assumed to increase linearly and uniformly (or remain constant) as the level of protection decreases or the degree of adverse weather increases, and these expenses are functions of a single parameter, the cost–loss ratio. Nevertheless, the usefulness of the model is exemplified by the fact that it provided the framework for the formulation of the ranked probability score (Epstein, 1969), a strictly proper scoring rule for probability forecasts of ordered variables that represents a natural extension of the

TABLE 2. The conditional probabilities  $p_{jl}$  ( $j, l = 1, 2, 3$ ) for two hypothetical forecasting systems.

Event	Observed event $W_j$		
	$W_1$	$W_2$	$W_3$
(a) Forecasting System A: $p_{jl}^A$			
$F_1$	0.46	0.36	0.18
$F_2$	0.12	0.64	0.24
$F_3$	0.03	0.12	0.85
(b) Forecasting System B: $p_{jl}^B$			
$F_1$	0.58	0.33	0.09
$F_2$	0.10	0.67	0.23
$F_3$	0.02	0.11	0.87

Brier score (Brier, 1950). Thus, it is not unreasonable to believe that this particular model might play the same role in  $N$ -action,  $N$ -event situations that the standard cost-loss ratio model plays in two-action, two-event situations.

Expressions have been derived for the expected expense and expected value associated with the use of different types of information in the context of this  $N$ -action,  $N$ -event situation. It should be noted that these expressions are based on an *ex ante* (or decision-analytic) approach to value-of-information assessment, whereas many previous studies of the value of meteorological information (e.g., Thompson, 1952, 1962; Thompson and Brier, 1955; Murphy, 1977) have relied on an *ex post* approach. Differences and similarities between the two approaches are briefly described in the Appendix. Three types of information have been considered in this paper: 1) climatological information; 2) perfect information; and 3) imperfect forecasts. Climatological and perfect information provide lower and upper bounds, respectively, on the quality of imperfect forecasts. Imperfect forecasts are taken here to be categorical forecasts properly calibrated according to past performance. In defining the value-of-information expressions employed in this paper, the expected expense associated with climatological information has been taken to represent the zero point on the value scale. Moreover, in this *ex ante* framework, the expected value of imperfect forecasts is nonnegative and the expected value of perfect information provides an upper bound on the expected value of all imperfect forecasts.

To illustrate this generalized model and the value of perfect and imperfect forecasts, we examined the three-action, three-event situation in some detail. Consideration of such a situation allowed certain results to be described in geometrical as well as analytical terms. Expressions for the value of perfect and imperfect forecasts in this situation were presented and numerical examples were used to illustrate the results. These examples included the evaluation of two hypothetical forecasting systems, and this process provided an opportunity to consider, albeit briefly, the relationship between the value and quality of imperfect forecasts. Obviously, it would be desirable to investigate quality/value relationships in this context in greater detail, including the use of both categorical and probabilistic measures of performance.

Many extensions of the generalized cost-loss ratio situation considered here are possible. For example,  $N$ -action,  $N$ -event models could be formulated with different definitions of the relevant costs and losses. As noted previously, the expenses considered in this paper increase linearly and uniformly (or remain constant) as the level of protection decreases and the degree of adverse weather increases. Alternative definitions of these expenses might be more appropriate for some classes of decision-making problems. More-

over, since many decision makers are risk averse, it might also be desirable to transform expenses into utilities via a risk-averse utility function and then to investigate the expected utility of different types of forecasts in this more realistic framework. Another possible extension would involve the consideration of more general types of imperfect forecasts such as probability forecasts. In effect, this extension simply involves including additional forecasts in the set of possible forecasts (i.e., removing the restriction that the number of distinct forecasts is equal to  $N$ ). Of course, it would be necessary to specify conditional probabilities (i.e.,  $p_{ji}$ ) for this "richer" set of possible forecasts. Finally, it would be desirable to formulate dynamic models for this and other  $N$ -action,  $N$ -event decision-making problems. The existence of such models would allow repetitive decision-making situations to be considered within this generalized framework. In addition to the formulation of the models themselves, it clearly would be desirable to investigate the value of different types of information within these various decision-making problems. Such studies would provide the meteorologist and the weather-information-sensitive decision maker with valuable tools for studying the optimal use and value of meteorological information in multiple-action, multiple-event situations.

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#### APPENDIX

##### Value-of-Information Assessment: *Ex Post* and *Ex Ante* Approaches

Studies of the value of forecasts in meteorology traditionally have been based on an *ex post* approach. This approach is concerned with determining the *actual* value of the forecasts of interest *after* the forecasts and relevant observations have become available. Thus, value-of-information estimates obtained using an *ex post* approach relate to a *specific* set of forecasts and observations. It should also be noted that the forecasts are taken at face value in the *ex post* approach—that is, it is assumed that users base their decisions on the likelihoods of occurrence of the events as specified by the forecasts, whether the latter are expressed in categorical or probabilistic terms (the implications of this statement will be clarified in an example below).

The *ex ante* approach, on the other hand, is concerned with determining the *expected* value of the forecasts *before* the forecasts and observations have actually become available. This approach is consistent with the methodology of decision analysis, and this

methodology has been employed extensively by economists, management scientists, operations researchers, and statisticians. In the *ex ante* approach, a statistical model is generally formulated that describes the joint distribution of the forecasts and observations that are expected to be provided to the decision maker. If attention is focused on the value of current forecasts, then it might be reasonable to base this model on an empirical distribution of forecasts and observations. However, it generally would not be appropriate to utilize a model that corresponds exactly to some specific set of forecasts and observations (due to sampling variability). Finally, it should be noted that the model describing the joint distribution of forecasts and observations provides a means of "calibrating" the forecasts, i.e., this model yields probabilities of the relevant events conditional on the forecasts (once again, whether the latter are categorical or probabilistic). Users are assumed to base their decisions on these calibrated forecasts.

To illustrate some important differences between the *ex post* and *ex ante* approaches and the respective value-of-information estimates, we briefly discuss a simple example. Consider a set of categorical forecasts of precipitation occurrence, and suppose that measurable precipitation never occurs when it is forecast to occur and that it always occurs when it is not forecast to occur. In the *ex post* approach, such "perverse" forecasts would be taken at face value (by the decision maker). Not surprisingly, then, the value of these forecasts would be very low—lower in fact than the value of forecasts based solely on long-term or sample climatological probabilities. In the *ex ante* approach, however, the perverse forecasts and the relevant observations would be modeled, yielding calibrated forecasts that presumably would prescribe a very high probability of precipitation when it is not forecast categorically, and a very low probability of precipitation when it is forecast categorically. Thus, the value of such forecasts would approach the value of perfect forecasts. As a consequence, the two approaches produce quite different results for this set of forecasts. Similar but less extreme differences in value between the two approaches would occur for less perverse sets of forecasts.

On the other hand, the *ex post* and *ex ante* value-of-information estimates would be approximately the same under certain conditions. These conditions include: 1) a large sample of forecasts and observations; 2) forecasts expressed in probabilistic terms; and 3) reliable forecasts (this condition implies that no calibration would be necessary). When such conditions hold, the *ex post* value can be viewed as an estimate of the *ex ante* value.

It is not the purpose of this appendix to discuss the relative merits of the *ex ante* and *ex post* approaches to value-of-information assessment. Nevertheless, if interest centers on assessing the value of forecasts before the latter are available, including the incremental value of future improvements in forecasts, then the *ex ante* approach would appear to be more appropriate than the *ex post* approach. Moreover, as noted above, the *ex ante* approach is consistent with decision-analytic methodology for analyzing and modeling the use and value of information, and such methodology has been employed in this paper.

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